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THE CHROMATIC NUMBER OF RANDOM GRAPHS AT THE DOUBLE-JUMP THRESHOLD

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A crucial step in the Erdős—Rényi (1960) proof that the double-jump threshold is also the planarity threshold for random graphs is shown to be invalid. We prove that when $p=1/n$, almost all graphs do not contain a cycle with a diagonal edge, contradicting Theorem 8a of Erdős and Rényi (1960). As a consequence, it is proved that the chromatic number is 3 for almost all graphs when $p=1/n$.

1. Introduction

The evolution of random graphs has been a topic of substantial interest since the seminal paper of Erdős and Rényi (1940). [For more complete discussions, see Bollobás (1985) and Palmer (1985).] A principal object of study is the random graph $K_{n,p}$, constructed on n labelled vertices with each of the $\binom{n}{2}$ possible edges present independently with probability p, $0 \leqq p \leqq 1$. The edge probability p is often permitted to vary as a function of n, with interest focusing on cases when $p=p(n)\to 0$ as $n\to\infty$. A property is said to hold for *almost all graphs* if the probability that the property is true for $K_{n,p}$ converges to one as $n\to\infty$. Many important results on the evolution of random graphs determine a "threshold function" $f(n)$ such that the character of almost all graphs is dramatically different when $p(n) \succ f(n)$ than when $p(n) \prec f(n)$.

A remarkable threshold occurs at $p(n)=1/n$. If $p(n)=c/n$ where $c<1$, almost all graphs have no component which contains more than one cycle, and the largest component is of the order $\log n$. When $c=1$, the order of the largest component jumps to approximately $n^{2/3}$. When $c>1$, it jumps again to the order of magnitude of n. Thus, $p(n)=1/n$ is the threshold for a dramatic "double-jump" in the size of the largest component.

It is widely accepted in the literature, based on an argument of Erdős and Rényi (1960), that $p(n)=1/n$ is also the threshold for nonplanarity of almost all random graphs. Erdős and Rényi actually stated their results in terms of the alternative random graph model $K_{n,M}$, in which M edges, with $M \leqq \binom{n}{2}$, are chosen

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randomly in the complete graph K_n. However, the models $K_{n,p}$ and $K_{n,M}$ are essentially equivalent, provided that M is near $p\binom{n}{2}$ [e.g., see Bollobás (1985), pp. 34—36]. For convenience, we state and prove results in terms of the $K_{n,p}$ model, but our methods apply to the $K_{n,M}$ model as well. The Erdős and Rényi nonplanarity result is based on the following claim:

Claim [Erdős and Rényi (1960 Theorem 8a]: *Let* $p(n)=(1+\lambda/\sqrt{n})/n$ *for a fixed* λ, *where* $-\infty<\lambda<\infty$. *Let* $X_n(d)$ *denote the number of cycles with exactly d diagonal edges in the random graph* $K_{n,p}$. *Then*

$$\lim_{n\to\infty} P[X_n(d)=i] = \frac{\varrho^i e^{-\varrho}}{i} \quad \forall i \geq 0,$$

where

$$\varrho \equiv \varrho(d,\lambda) = \frac{1}{2\cdot 6^d d!}\int_0^\infty t^{2d-1} e^{\lambda t/\sqrt{3}-t^2/2}\,dt.$$

The Claim states that the number of cycles with exactly d diagonal edges has an asymptotic Poisson distribution when $p(n)=(1+\lambda/\sqrt{n})/n$, and the mean of this distribution tends to $+\infty$ as $\lambda\to+\infty$. Observe that, when $d=3$, then approximately $1/15$ of the induced subgraphs with three diagonals are topological copies of the nonplanar bipartite graph $K_{3,3}$. Thus, if Theorem 8a is valid, by considering arbitrarily large λ, one would prove that almost all graphs are nonplanar when $c>1$. Moreover, setting $\lambda=0$ would provide the information that

$$\liminf_{n\to\infty} P[K_{n,p} \text{ is nonplanar}] > 0$$

at the threshold $p(n)=1/n$, as claimed by Erdős and Rényi (1960).

In this paper, we will demonstrate that the Erdős and Rényi proof of Theorem 8a is not valid, and that the Claim is actually incorrect. The method of proof they proposed is to show convergence of the factorial moments of the number of induced cycles with exactly d diagonal edges to the factorial moments of the asymptotic Poisson distribution. In Section 2, we provide an elementary argument to show that this method does not apply, since the second factorial moment diverges to $+\infty$.

We note, however, that $p(n)=1/n$ is in fact the threshold function for planarity, as a consequence of results of Ajtai, Komlós, and Szemerédi (1979, 1981). They proved that for any fixed r, almost all graphs contain a topological copy of K_r, the complete graph on r vertices, when $p(n)=c/n$ for $c>1$. The nonplanarity of the complete graph K_5 then implies that $p(n)=1/n$ is the nonplanarity threshold. The main result of this paper is the following:

Theorem. *If* $p(n)=1/n$, *then almost all graphs do not contain a cycle with at least one diagonal edge.*

Clearly, this result contradicts the existence of an asymptotic Poisson distribution for the number of cycles with exactly d diagonal edges when $\lambda=0$ in the Claim. After developing bounds for the number of connected sparsely edged graphs in Section 3, and preliminary lemmas regarding the structure of $K_{n,p}$ in Section 4, the Theorem will be proved in Section 5.

Erdős and Rényi (1960), § 10, noted that when $p(n)=1/n$, almost all graphs contain an odd cycle, and thus almost all graphs have chromatic number at least 3. Since Voss (1982) proved that every graph with chromatic number at least 4 contains as a subgraph either the complete graph K_4 or an odd circuit with at least two diagonal edges, we have the following:

Corollary. *If* $p(n)=1/n$, *then almost all graphs have chromatic number* 3.

In the remainder of the paper, unless otherwise indicated we intend the terms "graph", "subgraph", and "multigraph" to mean "labeled graph", "labeled subgraph", and "labeled multigraph", respectively.

2. The Flaw in the Claim's Proof

The proof of the Claim suggested by Erdős and Rényi is by the factorial moment method: They claimed convergence in distribution occurs since for each $k=1, 2, \ldots$, the k-th factorial moment $E_k[X_n(d)]$ converges to $p(d, \lambda)^k$ as $n\to\infty$. An elementary computation verifies this for $k=1$. Erdős and Rényi then suggest that the remainder of the proof follows as in their Theorem 2a, which proves Poisson convergence of the number of isolated trees. In the remainder of this section, for simplicity we consider only the case when $\lambda=0$, and show that $E_2[X_n(d)]\to\infty$ as $n\to\infty$. (All arguments can be repeated to show that this remains true for any constant λ.) Therefore, Poisson convergence of $X_n(d)$ cannot be proved by the convergence of factorial moments of all orders. Since factorial moment convergence is sufficient, but not necessary, for convergence to a Poisson distribution, the divergence of $E_2[X_n(d)]$ does not prove that the Claim is false, but only that the proof suggested is not valid. We prove in Section 5 that the Claim is false.

Let $\mathcal{D}_{d,n}$ denote the set of subgraphs of K_n which are cycles with exactly d diagonals. By a standard representation, $E_2[X_n(d)]$ is the expected number of subgraphs of $K_{n,p}$ which are the union of two distinct induced subgraphs from $\mathcal{D}_{d,n}$.

We will count a subclass of the set of pairs of induced cycles with exactly d diagonals. Let E_d denote the graph on the $4d+3$ vertices $v_0, v_1, v_2, x_1, \ldots, x_{2d}, y_1, \ldots, y_{2d}$ with the edges listed in the following two groups:

(i) (v_0, x_1), (v_2, v_0), (x_{2d}, v_2), (y_{2d}, v_2), (v_1, y_1), and (x_i, x_{i+1}) and (y_i, y_{i+1}) for each $i = 1, 2, \ldots, 2d-1$.

(ii) (v_0, v_1), and (x_i, x_{i+d}) and (y_i, y_{i+d}) for each $i = 1, 2, \ldots, d$.

A representation of E_d is given in Figure 1. The graph consists of a cycle with a diagonal edge, which creates two cycles, with d diagonals in each of these cycles (between vertices of the original cycle).

If $e_1, \ldots, e_k$ are edges of a graph G, *subdividing* $e_1, \ldots, e_k$ means replacing each of these edges by paths. A graph H is a *topological copy* of G if it is obtained by subdividing any set of edges of G.

Let $\mathcal{E}_{d,n}$ denote the set of subgraphs of K_n which are isomorphic to a graph obtained from E_d by subdividing any subset of the edges in group (i). An example of a graph in $\mathcal{E}_{d,n}$ is shown in Figure 2. The set of induced subgraphs of $K_{n,p}$ which are in $\mathcal{E}_{d,n}$ form a subclass of the pairwise unions of induced subgraphs in $\mathcal{D}_{d,n}$.

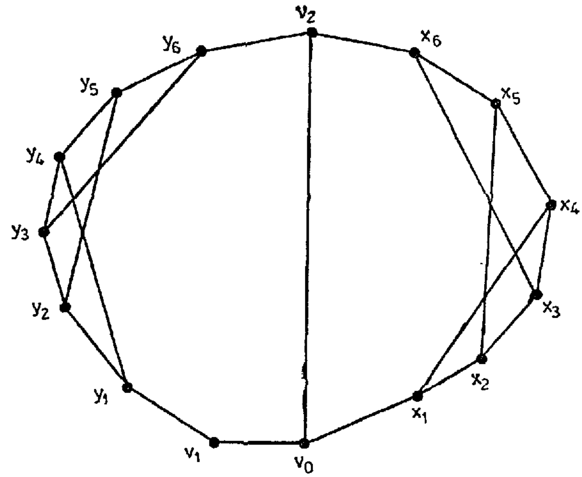

Fig. 1. A representation of E_3

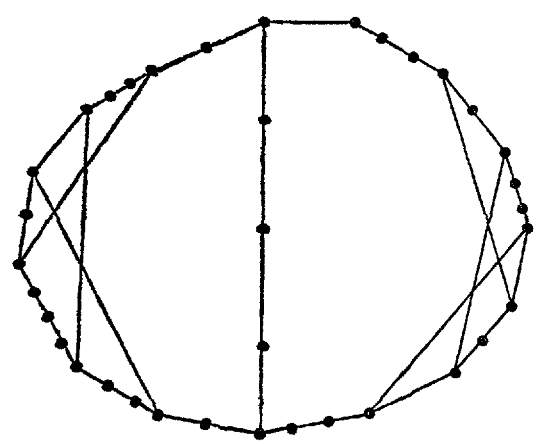

Fig. 2. A graph obtained from E_3 by subdividing a subset of the edges in group (i) in the definition of E_3

Letting $Y_n(d)$ denote the number of induced subgraphs from $\mathscr{E}_{d,n}$, we have

$$E_2[X_n(d)] \geqq E[Y_n(d)].$$

To construct a graph in $\mathscr{E}_{d,n}$ on a particular set of k vertices, first order the k vertices, then select $4d+2$ locations between vertices in this ordering to break the ordering into $4d+3$ ordered paths (some of which may consist of a single vertex). Use these paths to replace the edges of E_d in group (i), in the order listed, with the initial vertex of the path replacing the initial vertex of the edge as listed.

Since there are $k!$ orderings of the vertices, and $\binom{k-1}{4d+2}$ divisions of the ordering into paths, and $\binom{n}{k}$ possible sets of vertices, the number of graphs in

$\mathscr{E}_{d,n}$ on k vertices is

$$\binom{n}{k}k!\binom{k-1}{4d+2}.$$

Each of these graphs is present as an induced subgraph in $K_{n,p}$ with probability

$$p^{k+2d+1}(1-p)^{\binom{k}{2}-k-2d-1}.$$

We may now compute a lower bound for $E[Y_n(d)]$. In the following (and throughout the remainder of the paper) A denotes a constant independent of n, whose value may vary from appearance to appearance in a sequence of inequalities.

$$E[Y_n(d)] = \sum_{k=4d+2}^{n}\binom{n}{k}k!\binom{k-1}{4d+2}p^{k+2d+1}(1-p)^{\binom{k}{2}-k-2d-1} >$$

$$> An^{-2d-1}\sum_{k=n^{1/2}}^{2n^{1/2}}\frac{n(n-1)\dots(n-k+1)}{n^k}k^{4d+2}\left(1-\frac{1}{n}\right)^{\binom{k}{2}}$$

which, by absorbing lower bounds for the other factors into the constant A, is

$$> An^{-2d-1}\sum_{k=n^{1/2}}^{2n^{1/2}}k^{4d+2} > An^{1/2}\to\infty.$$

Thus, as we claimed, the second factorial moment diverges as $n\to\infty$, and the proof suggested by Erdős and Rényi is not valid.

3. Sparse Graph Counts

Our proof relies on a bound for the number of graphs with specified numbers of vertices and edges which contain a cycle with diagonal edges. In this section, we modify the method of Bollobás (1984) to provide an appropriate bound.

Let $C(k, k+l)$ denote the number of connected graphs on n labeled vertices which have exactly k vertices and $k+l$ edges. Since connectedness is required, clearly $C(k, k+l)=0$ for $l<-1$. An elementary upper bound for $C(k, k+l)$, valid for all $0\leq l\leq\binom{k}{2}-k$, is

$$C(k,k+l)\leq\binom{\binom{k}{2}}{k+l}\leq\left(\frac{ek^2}{2(k+l)}\right)^{k+l}. \tag{3.1}$$

This bound will be applied in cases when l is large. More precise bounds are needed for cases when l is small. The number of trees was evaluated by Cayley (1889), obtaining

$$C(k,k-1)=k^{k-2}. \tag{3.2}$$

An exact expression for $C(k, k)$ was determined by Katz (1955) and Rényi (1959), using the fact that the number of forests on $\{1, 2, \dots, r\}$ which have s components and in which the vertices $1, 2, \dots, s$ belong to distinct components is

$$sr^{r-1-s}. \tag{3.3}$$

A very sophisticated method of Wright (1980) gave an estimate for $C(k, k+l)$, which is valid up to a factor of $1+o(1)$ for $l=o(k^{1/3})$. Bollobás (1984) observed that the following uniform bound on $C(k, k+l)$ can be obtained by using much simpler arguments.

Lemma 3.4. *There exists an absolute constant c such that for* $1 \leqq l \leqq k$,

$$C(k, k+l) \leqq \left(\frac{c}{l}\right)^{l/2} k^{k+(3l-1)/2}.$$

Since our result is obtained by a slight modification of the Bollobás result, we now provide a sketch of the proof of Lemma 3.4, and later describe alterations to obtain our Lemma 3.6.

Sketch of Bollobás' construction. Assume that $l \geqq 1$, and let G be a connected graph with k vertices and $k+l$ edges. G contains at least 2 cycles, so it contains a unique maximal connected subgraph G^{I} with minimum degree at least 2, obtained by removing a forest with roots on the cycles of G. G^{I} has exactly l more edges than vertices.

We construct a connected multigraph G^{II} with minimum degree at least 3 and exactly l more edges than vertices, by removing all vertices of degree 2 in G^{I}. [We define the degree of vertex v of a multigraph as $2d_0(v)+d_1(v)$, where $d_0(v)$ is the number of loops which are incident to the vertex v, and $d_1(v)$ is the number of other edges which are incident to v.] The condition that the minimum degree of G^{II} is at least 3, and the fact that G^{II} has exactly l more edges than vertices, together imply that G^{II} has at most $2l$ vertices.

Such a multigraph on t vertices, $t \leqq 2l$, is determined by the partition of its $t+l$ edges and loops into $\binom{t}{2}+t$ classes for (multiple) edges and (multiple) loops. Thus, there are at most

$$\binom{k}{t}\binom{\binom{t}{2}+t+(t+l-1)}{t+l}$$

such multigraphs G^{II} on t vertices chosen from k vertices.

For some u, with $1 \leqq u \leqq k-t$, u vertices may be inserted on the edges of G^{II} to obtain a graph G^{I} in at most

$$u!\binom{u+t+l-1}{t+l-1} \tag{3.5}$$

ways.

From the remaining $k-t-u$ vertices, a forest to attach to the $t+u$ vertices of G^{I} may be constructed in at most

$$(t+u)k^{k-1-t-u}$$

ways, by (3.3).

Combining these estimates yield the bound:

$$C(k, k+l) \leqq \sum_{t=1}^{2l} \binom{k}{t}\binom{\binom{t}{2}+2t+l-1}{t+l} \sum_{u=0}^{k-t}\binom{k-t}{u} u!\binom{u+t+l-1}{t+l-1}(t+u)k^{k-1-t-u}.$$

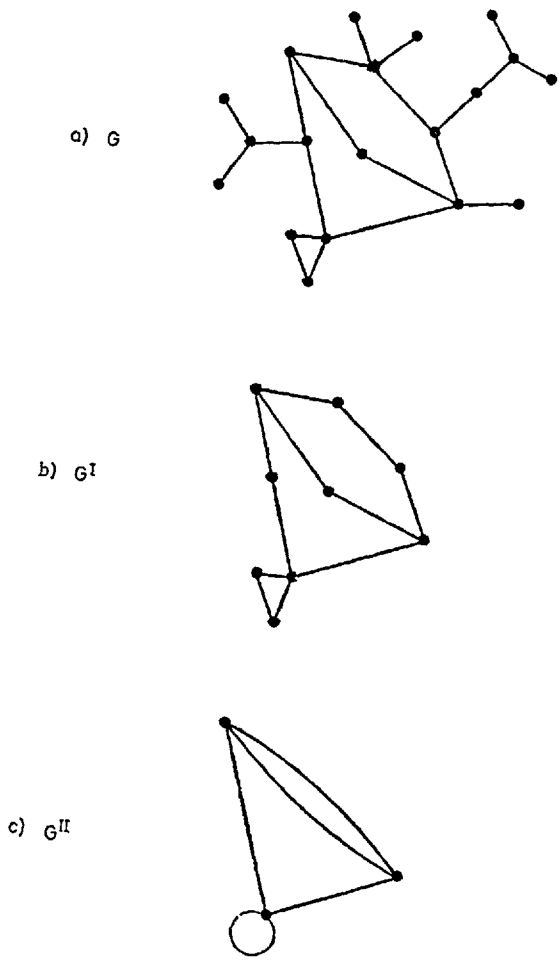

Fig. 3. An illustration of the Bollobás construction

For computations to obtain the bound in Lemma 3.4 from this expression, see Bollobás (1984). ∎

Let $D(k, k+l)$ denote the number of connected graphs with k vertices and $k+l$ edges which contain at least one cycle with at least one diagonal edge.

Lemma 3.6. *There exists an absolute constant d such that for* $1 \leqq l \leqq k$,

$$D(k, k+l) = \left(\frac{d}{l}\right)^{l/2} k^{k+\frac{3l}{2}-1}.$$

Proof. The essential difference from Bollobás construction in Lemma 3.4 is that one must choose at least one edge of G^{II} to receive no inserted vertices, before inserting u vertices in the multigraph G^{II} to obtain G^{I}. Therefore, expression (3.5) is replaced by

$$u!\left(\binom{t}{2}+t\right)\binom{u+t+l-2}{t+l-2},$$

which leads to the lower exponent than in Lemma 3.4. The remainder of the proof follows Bollobás' proof closely. ∎

4. Structure Lemmas

This section establishes two facts about the size and structure of components of $K_{n,p}$ when $p(n)=1/n$.

Lemma 4.1. *Let $p(n)=1/n$, $u(n)\to 0$ and $w(n)\to+\infty$ as $n\to\infty$. Then in almost all random graphs, all components which have more edges than vertices have between $u(n)n^{2/3}$ and $w(n)n^{2/3}$ vertices.*

Proof. It is well-known [see Erdős and Rényi (1960)] that when $p(n)=1/n$, the largest component has fewer than $w(n)n^{2/3}$ vertices in almost all graphs.

To show that components with more edges than vertices cannot have fewer than $u(n)n^{2/3}$ vertices, we proceed by the first moment method. Let $X_n(k,k+l)$ denote the number of components of $K_{n,p}$ which have exactly k vertices and $k+l$ edges. We bound

$$P\left(\sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{u(n)n^{2/3}} X_n(k,k+l)>0\right)\leqq \sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{u(n)n^{2/3}} E[X_n(k,k+l)]=$$

$$=\sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{u(n)n^{2/3}}\binom{n}{k}C(k,k+l)p^{k+l}(1-p)^{nk-\binom{k}{2}+k+l}.$$

It is necessary to split the sum into two parts, depending on the value of k, due to the restriction in Lemma 3.5 that $1\leqq l\leqq k$.

For large l, we apply (3.1) to obtain

$$=\sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{\max\{l,u(n)n^{2/3}\}}\binom{n}{k}\left(\frac{ek^2}{2(k+l)}\right)^{k+l}p^{k+l}(1-p)^{nk-\binom{k}{2}+k+l}\leqq$$

$$\leqq A\sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{\max\{l,u(n)n^{2/3}\}}\frac{1}{k!}e^{-k^2/2n-k^3/6n^2}\left(\frac{ek^2}{2(k+l)}\right)^{k+l}n^{-l}e^{-k+\frac{k^2}{2n}}$$

which, by Stirling formula bounds for $k!$ and $\left(\frac{e}{2}\frac{k}{k+l}\right)<1$,

$$\leqq A\sum_{l=1}^{\binom{n}{2}}\sum_{k=1}^{\max\{l,u(n)n^{2/3}\}}\left(\frac{k}{n}\right)^l\leqq A\sum_{l=1}^{\infty}l\left(\frac{u(n)}{n^{1/3}}\right)^l$$

which converges to zero as $n\to\infty$.

For $1 \leqq l \leqq k$, we apply the bound of Lemma 3.5:

$$\sum_{l=1}^{\binom{n}{2}} \sum_{k=l}^{u(n)n^{2/3}} \binom{n}{k} C(k, k+l) p^{k+l} (1-p)^{nk-\binom{k}{2}+k+l} \leqq$$

$$\leqq A \sum_{l=1}^{\binom{n}{2}} \sum_{k=l}^{u(n)n^{2/3}} \frac{1}{k!} e^{-k^2/2n - k^3/6n^2} \left(\frac{c}{l}\right)^{l/2} k^{k+\frac{3l-1}{2}} n^{-l} e^{-k+\frac{k^2}{2n}} \leqq$$

$$\leqq A \sum_{l=1}^{\binom{n}{2}} \sum_{k=l}^{u(n)n^{2/3}} \frac{e^{-k}}{k!} \left(\frac{c}{l}\right)^{l/2} n^{-l} k^{k+\frac{3l-1}{2}}$$

which, by Stirling formula bounds for $k!$,

$$\leqq A \sum_{l=1}^{\binom{n}{2}} \left(\frac{c}{l}\right)^{l/2} n^{-l} \sum_{k=1}^{u(n)n^{2/3}} k^{\frac{3l-2}{2}} \leqq A \sum_{l=1}^{\binom{n}{2}} \left(\frac{c}{l}\right)^{l/2} \frac{2}{3l} u(n)^{3l/2}$$

which converges to zero as $n \to \infty$. ∎

Lemma 4.2. *Let $p(n) = 1/n$ and $z(n) \to \infty$. Then in almost all graphs, all components have fewer than $z(n)$ more edges than vertices.*

Proof. We proceed by the first moment method, using Lemma 4.1 to restrict the range of summation for k:

$$P\left(\sum_{l=z(n)}^{\binom{n}{2}} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} X_n(k, k+l) > 0\right) \leqq \sum_{l=z(n)}^{\binom{n}{2}} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} E[X_n(k, k+l)] =$$

$$= \sum_{l=z(n)}^{\binom{n}{2}} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} \binom{n}{k} C(k, k+l) p^{k+l} (1-p)^{nk-\binom{k}{2}+k+l}.$$

As in the proof of Lemma 4.1, split the sum into parts, depending on the value of k.

For $k \leqq l$, the previous method applies, using (3.1) to obtain

$$= \sum_{l=z(n)}^{\binom{n}{2}} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} \binom{n}{k} C(k, k+l) p^{k+l} (1-p)^{nk-\binom{k}{2}+k+l} \leqq A \sum_{l=z(n)}^{\infty} l[w(n)n^{-1/3}]^l.$$

By choosing $\{w(n)\}$ such that $w(n)n^{-1/3} \to 0$ as $n \to \infty$, the sum converges to zero as $n \to \infty$.

For the other sum, we apply Bollobás bound from Lemma 3.5 as in the proof of Lemma 4.1, to obtain

$$= \sum_{l=z(n)}^{\binom{n}{2}} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} \binom{n}{k} C(k, k+l) p^{k+l} (1-p)^{k(n-k)-\binom{k}{2}+k+l} \leqq \tag{4.3}$$

$$\leqq \sum_{l=z(n)}^{\binom{n}{2}} \left(\frac{c}{l}\right)^{l/2} \sum_{k=0}^{w(n)n^{2/3}} k^{\frac{3l-2}{2}} n^{-l} e^{-k^3/6n^2}$$

so, bound the exponential factor by one, then multiply the maximum term in the inner sum by the number of terms, to obtain

$$\leqq A \sum_{l=z(n)}^{\binom{n}{2}} \left(\frac{c}{l}\right)^{l/2} w(n)^{3l/2} \leqq A \sum_{l=z(n)}^{n^{2/3}} \left[\frac{cw(n)^3}{z(n)}\right]^{l/2}.$$

By choosing $\{w(n)\}$ such that $w(n)^3/z(n) \to 0$, this bound converges to zero as $n \to \infty$. ∎

5. Proof of Theorem

By Lemma 4.1, we need only consider components with approximately $n^{2/3}$ vertices. By Lemma 4.2, we may neglect components which have more than $z(n)$ more edges than vertices, where $z(n) \to \infty$ arbitrarily slowly. We now apply the first moment method to the number of components containing a cycle with at least one diagonal edge, which have between $u(n)n^{2/3}$ and $w(n)n^{2/3}$ vertices $(u(n) \to 0,\ w(n) \to +\infty)$ and have less than $z(n)$ more edges than vertices $(z(n) \to \infty)$. The expected number of appropriate components is

$$\sum_{l=1}^{z(n)} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} \binom{n}{k} D(k, k+l) p^{k+l} (1-p)^{k(n-k)+\binom{k}{2}-k-l}.$$

Replace $D(k, k+l)$ by the bound from Lemma 3.6, p by $1/n$, use $\left(1-\frac{1}{n}\right) \leqq e^{-1/n}$, and rearrange factors to obtain

$$\sum_{l=1}^{z(n)} \frac{1}{n^l} \left(\frac{d}{l}\right)^{l/2} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} \frac{(n)_k}{n^k} \frac{k^{k+\frac{3l}{2}-1}}{k!} e^{-\frac{1}{n}\left[k(n-k)+\binom{k}{2}-k-l\right]}.$$

Simplify the exponent on e to $-k+\dfrac{k^2}{2n}+\dfrac{3k}{2n}+\dfrac{l}{n}$, and note that $k/n \leqq 1$ and $l/n \leqq z(n)/n \leqq 1$. From this, $(n)_k/n^k \leqq A \exp\{-k^2/2n - k^3/6n^2\}$, and the fact that Stirling's formula is a lower bound for $k!$, we obtain

$$A \sum_{l=1}^{z(n)} \frac{1}{n^l} \left(\frac{d}{l}\right)^{l/2} \sum_{k=u(n)n^{2/3}}^{w(n)n^{2/3}} e^{-k^2/2n - k^3/6n^2} k^{k+3l/2-1} (e^k k^{-k-1/2}) e^{-k+k^2/2n} \leqq$$

$$\leqq A \sum_{l=1}^{z(n)} \frac{1}{n^l} \left(\frac{d}{l}\right)^{l/2} \sum_{k=0}^{w(n)n^{2/3}} e^{-k^3/6n^2} k^{3l/2-3/2}.$$

Bound the exponential factor by one, and replace the inner sum by the maximum term times the number of terms, to obtain

$$\leqq \frac{A}{n^{1/3}} \sum_{l=1}^{z(n)} d^{l/2} w(n)^{3l/2-1/2} \leqq \frac{A}{n^{1/3}} \sum_{l=1}^{z(n)} [dw(n)^3]^{l/2} \leqq \frac{A}{n^{1/3}} [dw(n)^3]^{z(n)/2}$$

which converges to zero if $z(n)$ and $w(n)$ are both $o(\log\log n)$, for example. Thus, by the first moment method, almost all graphs contain no appropriate components, completing the proof for the $K_{n,p}$ model. By Bollobás (1985) Theorem 2, the conclusion of the Theorem also holds for the random graph model $K_{n,M}$, where $M=\frac{n-1}{2}$, which was the setting of Theorem 8a of Erdős and Rényi (1960). ∎

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